

# BEURLING'S CRITERION AND EXTREMAL METRICS FOR FUGLEDE MODULUS

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**ABSTRACT.** For each  $1 \leq p < \infty$ , we formulate a necessary and sufficient condition for an admissible metric to be extremal for the Fuglede  $p$ -modulus of a system of measures. When  $p = 2$ , this characterization generalizes Beurling's criterion, a sufficient condition for an admissible metric to be extremal for the extremal length of a planar curve family. In addition, we prove that every Borel function  $\varphi : \mathbb{R}^n \rightarrow [0, \infty]$  satisfying  $0 < \int \varphi^p < \infty$  is extremal for the  $p$ -modulus of some curve family in  $\mathbb{R}^n$ .

## 1. INTRODUCTION

In this note we take a close look at extremal metrics for systems of measures and families of curves. Let us start by recalling Fuglede's definition of modulus [6]. Fix once and for all a measure space  $(X, \mathcal{M}, m)$ . A collection of measures  $\mathbf{E}$  is a *measure system* (over  $\mathcal{M}$ ) if each measure  $\mu \in \mathbf{E}$  is defined on the  $\sigma$ -algebra  $\mathcal{M}$ . A Borel function  $\varphi : X \rightarrow [0, \infty]$  is called a *metric* and is said to be *admissible* for  $\mathbf{E}$  if  $\int \varphi d\mu \geq 1$  for all  $\mu \in \mathbf{E}$ . (We do not identify two metrics which agree  $m$ -a.e.) For each  $0 < p < \infty$ , the  $p$ -modulus of  $\mathbf{E}$  is given by

$$\text{mod}_p \mathbf{E} = \inf \left\{ \int \varphi^p dm : \varphi \text{ is admissible for } \mathbf{E} \right\}$$

where  $\text{mod}_p \mathbf{E} = \infty$  if admissible metrics for  $\mathbf{E}$  do not exist.

*Example 1.* To pick a concrete setting, take  $(X, \mathcal{M}, m) = (\mathbb{R}^n, \mathcal{B}_n, m_n)$  where  $m_n$  is Lebesgue measure on the Borel subsets  $\mathcal{B}_n$  of  $\mathbb{R}^n$ . A (locally rectifiable) curve  $\gamma$  in  $\mathbb{R}^n$  is a concatenation of countably many images of one-to-one Lipschitz maps  $\gamma_i : [a_i, b_i] \rightarrow \mathbb{R}^n$ . Each image  $\gamma_i([a_i, b_i])$  is called a *piece* of  $\gamma$ ; curves may have disjoint or overlapping pieces. (For an alternative definition of a curve, see [17].) The *trace* of a curve  $\gamma$  is the set  $\bigcup_i \gamma_i([a_i, b_i])$  that is the union of the pieces of  $\gamma$ . For every curve  $\gamma$  in  $\mathbb{R}^n$  there is a Borel measure  $\tilde{\gamma}$  on  $\mathbb{R}^n$  such that the line integral

$$\int_{\gamma} f ds = \sum_i \int_{a_i}^{b_i} f(\gamma_i(t)) |\gamma_i'(t)| dt$$

is given by integration against  $\tilde{\gamma}$ , i.e.  $\int_{\gamma} f ds = \int_{\mathbb{R}^n} f d\tilde{\gamma}$  for every Borel function  $f$ . (By the area formula  $\tilde{\gamma} = \sum_i \tilde{\gamma}_i$  where  $\tilde{\gamma}_i = H^1 \llcorner \gamma_i([a_i, b_i])$  is the 1-dimensional Hausdorff measure restricted to the set  $\gamma_i([a_i, b_i])$ , e.g. see [5].) For all  $1 \leq p < \infty$ , the  $p$ -modulus of a family of curves  $\Gamma$  in  $\mathbb{R}^n$  is defined in terms of Fuglede modulus

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by  $\text{mod}_p \Gamma = \text{mod}_p \{\tilde{\gamma} : \gamma \in \Gamma\}$ . Therefore,  $\text{mod}_p \Gamma = \inf_{\varphi} \int_{\mathbb{R}^n} \varphi^p (dm_n)$  where the infimum runs over all Borel functions  $\varphi \geq 0$  such that  $\int_{\gamma} \varphi ds \geq 1$  for all  $\gamma \in \Gamma$ . (One could similarly define  $\text{mod}_p \Gamma$  for  $0 < p < 1$ , but this quantity is always zero.) In the plane, the *extremal length*  $\lambda(\Gamma) = 1/\text{mod}_2 \Gamma$  of a curve family  $\Gamma$  in  $\mathbb{R}^2$  is often used instead of its modulus.

An *atom* in a  $\sigma$ -algebra  $\mathcal{M}$  is a nonempty set  $A \in \mathcal{M}$  with the property  $B \subsetneq A$ ,  $B \in \mathcal{M} \Rightarrow B = \emptyset$ . That is, the only proper measurable subset of an atom is the empty set. If  $\varphi : X \rightarrow [0, \infty]$  is a Borel function on  $(X, \mathcal{M})$ , then  $\varphi$  is constant on each atom of  $\mathcal{M}$ . Given an atom  $A \in \mathcal{M}$ , the *atomic measure*  $\delta_A$  is defined by the rule  $\delta_A(S) = 1$  if  $A \subset S$  and  $\delta_A(S) = 0$  otherwise;  $\int \varphi d\delta_A = \varphi(A)$  for all  $\varphi$  and  $A$ .

*Example 2.* Let  $\mathcal{K} = \{K_1, \dots, K_\ell\} \subset \hat{\mathbb{C}}$  be a finite set of pairwise disjoint compact subsets of the Riemann sphere  $\hat{\mathbb{C}}$ , and let  $\Omega \subset \hat{\mathbb{C}}$  be an open set. The *transboundary measure space*  $(\hat{\mathbb{C}}, \mathcal{M}_{\mathcal{K}}, m_{\Omega, \mathcal{K}})$  is defined as follows. Let  $\mathcal{B}(\hat{\mathbb{C}} \setminus K)$  denote the Borel  $\sigma$ -algebra on the complement of  $K = \bigcup_{i=1}^{\ell} K_i$ . Then  $\mathcal{M}_{\mathcal{K}}$  is the smallest  $\sigma$ -algebra generated by  $\mathcal{B}(\hat{\mathbb{C}} \setminus K) \cup \mathcal{K}$ . The atoms of  $\mathcal{M}_{\mathcal{K}}$  are the singletons  $\{x\}$  with  $x \in \hat{\mathbb{C}} \setminus K$  and the sets  $K_1, \dots, K_\ell$ . We define the measure  $m_{\Omega, \mathcal{K}} = H^2 \llcorner (\Omega \setminus K) + \sum_{i=1}^{\ell} \delta_{K_i}$  where  $H^2 \llcorner (\Omega \setminus K)$  is 2-dimensional Hausdorff measure on  $\Omega \setminus K$ . Let  $\gamma : [a, b] \rightarrow \hat{\mathbb{C}}$  be a one-to-one continuous map, let  $\text{Im } \gamma = \gamma([a, b])$  be its image, and assume that  $\text{Im } \gamma \cap (\Omega \setminus K)$  is locally rectifiable. Then we define a measure  $\hat{\gamma}$  on  $(\hat{\mathbb{C}}, \mathcal{M}_{\mathcal{K}})$  by

$$\hat{\gamma} = H^1 \llcorner \text{Im } \gamma \cap (\Omega \setminus K) + \sum_{i: \text{Im } \gamma \cap K_i \neq \emptyset} \delta_{K_i}.$$

Suppose that  $(X, \mathcal{M}, m) = (\hat{\mathbb{C}}, \mathcal{M}_{\mathcal{K}}, m_{\Omega, \mathcal{K}})$ . The *transboundary modulus*  $\text{mod}_{\Omega, \mathcal{K}} \Gamma$  of a collection  $\Gamma$  of one-to-one continuous maps  $\gamma : [a, b] \rightarrow \hat{\mathbb{C}}$  is defined via Fuglede modulus by  $\text{mod}_{\Omega, \mathcal{K}} \Gamma = \text{mod}_2 \{\hat{\gamma} : \gamma \in \Gamma \text{ and } \text{Im } \gamma \cap (\Omega \setminus K) \text{ is locally rectifiable}\}$ . Thus, an admissible metric  $\varphi : \hat{\mathbb{C}} \rightarrow [0, \infty]$  satisfies

$$\int_{\text{Im } \gamma \cap (\Omega \setminus K)} \varphi ds + \sum_{i: \text{Im } \gamma \cap K_i \neq \emptyset} \varphi(K_i) \geq 1$$

for every  $\gamma \in \Gamma$  such that  $\text{Im } \gamma \cap (\Omega \setminus K)$  is locally rectifiable, and

$$\text{mod}_{\Omega, \mathcal{K}} \Gamma = \inf_{\varphi} \int_{\Omega \setminus K} \varphi^2 dH^2 + \sum_{i=1}^{\ell} \varphi(K_i)^2$$

where the infimum runs over all admissible metrics  $\varphi : \hat{\mathbb{C}} \rightarrow [0, \infty]$ . The reciprocal  $\lambda_{\Omega, \mathcal{K}}(\Gamma) = 1/\text{mod}_{\Omega, \mathcal{K}} \Gamma$  of transboundary modulus is *transboundary extremal length*.

*Remark 1.* The definition of extremal length is due to Beurling and has roots in the length-area principle for conformal maps; see Jenkins [12] for a historical overview. Since the introduction of extremal length by Ahlfors and Beurling [2], the modulus of a curve family has become a widely-used tool, employed in geometric function theory [1, 7, 15], quasiconformal and quasiregular mappings [17, 18], dynamical systems [8, 13], and analysis on metric spaces [10, 11]. The transboundary extremal length of a curve family was introduced by Schramm [16] to study uniformization on countably-connected domains. Recently Bonk [4] used transboundary modulus in a crucial way to obtain uniformization results on Sierpiński carpets in the plane. For applications of modulus of measures, see Fuglede's original applications in [6],

Hakobyan's work on the conformal dimension of sets [9], and Bishop and Hakobyan's recent paper on the frequency of dimension distortion by quasimetric maps [3].

A few nice properties of modulus are apparent from the definition. First if  $\mathbf{E} \subset \mathbf{F}$  then  $\text{mod}_p \mathbf{E} \leq \text{mod}_p \mathbf{F}$ . Second  $\text{mod}_p \bigcup_{i=1}^{\infty} \mathbf{E}_i \leq \sum_{i=1}^{\infty} \text{mod}_p \mathbf{E}_i$  for any sequence of measure systems. Since  $\text{mod}_p \emptyset = 0$ , this says that modulus is an outer measure on measure systems. A third useful property is that every admissible metric gives an upper bound on modulus, i.e.  $\text{mod}_p \mathbf{E} \leq \int \varphi^p dm$  for all admissible metrics  $\varphi$ .

If the infimum in the definition of the modulus of a measure system  $\mathbf{E}$  is obtained by an admissible metric  $\varphi$ , i.e. if  $\text{mod}_p \mathbf{E} = \int \varphi^p dm$ , then the metric  $\varphi$  is said to be *extremal* for the  $p$ -modulus of  $\mathbf{E}$ . Naturally one may ask whether an extremal metric always exists, and if so, to what extent is an extremal metric uniquely determined. Unfortunately simple examples (see Example 5 below) show that the existence and uniqueness of extremal metrics fails for general measure systems. Nevertheless, Fuglede [6] proved that when  $1 < p < \infty$ , a measure system always admits an extremal metric, after removing an exceptional system of measures.

**Fuglede's Lemma.** *Let  $1 < p < \infty$ . Let  $\mathbf{E}$  be a measure system. If  $\text{mod}_p \mathbf{E} < \infty$ , then there exists a measure system  $\mathbf{N} \subset \mathbf{E}$  such that  $\text{mod}_p \mathbf{N} = 0$  and  $\mathbf{E} \setminus \mathbf{N}$  admits an extremal metric  $\varphi$ .*

The uniqueness of an extremal metric for the  $p$ -modulus of a measure system also holds when  $1 < p < \infty$ , up to redefinition of the metric on a set of  $m$ -measure zero. This can be seen as follows. Suppose that  $\varphi, \psi \in L^p(m)$  are two extremal metrics for the  $p$ -modulus of a measure system  $\mathbf{E}$ . Then the averaged metric  $\chi = \frac{1}{2}\varphi + \frac{1}{2}\psi$  is still admissible for  $\mathbf{E}$  and  $(\text{mod}_p \mathbf{E})^{1/p} \leq \|\chi\|_p \leq \frac{1}{2}\|\varphi\|_p + \frac{1}{2}\|\psi\|_p = (\text{mod}_p \mathbf{E})^{1/p}$ . Thus,  $\|\frac{1}{2}\varphi + \frac{1}{2}\psi\|_p = \frac{1}{2}\|\varphi\|_p + \frac{1}{2}\|\psi\|_p$ . By the condition for equality in Minkowski's inequality and the assumption that  $\|\varphi\|_p = \|\psi\|_p < \infty$ , it follows that  $\varphi = \psi$   $m$ -a.e., as desired.

A fundamental problem working with modulus is to identify an extremal metric for a given measure system or curve family if one exists. Beurling found a general sufficient condition which guarantees that an admissible metric for a curve family in the plane is extremal for its extremal length.

**Beurling's Criterion** (Ahlfors [1], Theorem 4.4). *Let  $\Gamma$  be a curve family in  $\mathbb{R}^2$  and let  $\varphi$  be an admissible metric for  $\Gamma$  such that  $0 < \int_{\mathbb{R}^2} \varphi^2 < \infty$ . Suppose that there exists a curve family  $\Gamma_0$  in  $\mathbb{R}^2$  such that*

- (1)  $\Gamma_0 \subset \Gamma$ ,
- (2)  $\int_{\gamma} \varphi ds = 1$  for every  $\gamma \in \Gamma_0$ , and
- (3) for all  $f \in L^2(\mathbb{R}^2)$  taking values in  $[-\infty, \infty]$ : if  $\int_{\gamma} f ds \geq 0$  for all  $\gamma \in \Gamma_0$  then  $\int_{\mathbb{R}^2} f \varphi \geq 0$ .

*Then  $\varphi$  is an extremal metric for the extremal length of  $\Gamma$ , i.e.  $\lambda(\Gamma) = (\int_{\mathbb{R}^2} \varphi^2)^{-1}$ .*

Let us see Beurling's criterion in action, in a standard example.

**Example 3.** Let  $R$  be a rectangle with side lengths  $a \leq b$ . Let  $\Gamma$  be the family of all curves in  $R$  with connected trace which join opposite edges in  $R$  (see Figure 1). We claim that  $\varphi = \frac{1}{a}\chi_R$  is an extremal metric for  $\Gamma$ , and thus,

$$\lambda(\Gamma) = \left( \int_R \frac{1}{a^2} \right)^{-1} = a/b.$$

FIGURE 1. Curve families  $\Gamma$  and  $\Gamma_0$  in Examples 3 and 4

First  $\varphi$  is admissible for  $\Gamma$ , because every curve connecting opposite edges in  $R$  travels at least Euclidean distance  $a$  (the distance between the edges of length  $b$ ). Beurling's criterion holds with  $\Gamma_0$  equal to the family of straight line segments connecting opposite sides of length  $b$ . Conditions (1) and (2) hold by definition. Condition (3) follows from Fubini's theorem: if  $\int_{\gamma} f ds \geq 0$  for all  $\gamma \in \Gamma_0$ , then  $\int_{\mathbb{R}^2} f \varphi = \frac{1}{a} \int_R f = \frac{1}{a} \int_0^b \int_{\gamma(t)} f ds dt \geq 0$ . Therefore,  $\varphi$  is extremal for  $\lambda(\Gamma)$ .

The converse to Beurling's criterion fails for the simple reason that  $\Gamma$  may not contain any curves  $\gamma$  such that  $\int_{\gamma} \varphi ds = 1$ .

*Example 4.* Once again let  $R$  be a rectangle with side lengths  $a \leq b$ , and let  $\Gamma$  and  $\Gamma_0$  be the curve families from Example 3. We claim that  $\varphi = \frac{1}{a} \chi_R$  is an extremal metric for  $\Gamma_* = \Gamma \setminus \Gamma_0$ . However, since  $\Gamma_*$  does not contain any curves  $\gamma$  such that  $\int_{\gamma} \varphi ds = 1$ , we cannot use Beurling's criterion to show that  $\varphi$  is extremal for  $\Gamma_*$ . Let  $\psi$  be an admissible metric for  $\Gamma_*$ . Fix  $\gamma \in \Gamma_0$ . Then one can find a sequence of curves  $\gamma^k \in \Gamma_*$  such that  $\int_{\gamma^k} \psi ds \rightarrow \int_{\gamma} \psi ds$ . (For example, if  $\gamma = [0, a]$ , then take  $\gamma^k = [0, 1/k] \sqcup [0, a]$  where  $\sqcup$  denotes concatenation.) In particular, it follows that  $\int_{\gamma} \psi ds \geq 1$  for all  $\gamma \in \Gamma_0$ . Integrating across all  $\gamma \in \Gamma_0$ , invoking Fubini's theorem and applying the Cauchy-Schwarz inequality gives

$$b \leq \int_0^b \int_{\gamma(t)} \psi ds dt = \int_R \psi \leq \left( \int_R \psi^2 \right)^{1/2} (ab)^{1/2} \leq \left( \int_{\mathbb{R}^2} \psi^2 \right)^{1/2} (ab)^{1/2}.$$

Thus,  $(\int_{\mathbb{R}^2} \psi^2)^{-1} \leq a/b$  for every metric  $\psi$  that is admissible for  $\Gamma_*$ . Since this upper bound is obtained by  $\varphi$ , we conclude that  $\varphi$  is extremal for  $\lambda(\Gamma_*)$ .

*Remark 2.* A partial converse to Beurling's criterion is given in Ohtsuka [14, §2.3] in the special case  $\Gamma = \Gamma_0$ : if  $\varphi$  is extremal for  $\Gamma$ , then (3) holds for all  $f \in L^2(\mathbb{R}^2)$ . Wolf and Zwiebach [19, p. 38] have also established “a partial local converse to Beurling's criterion” for certain classes of metrics on Riemann surfaces.

## 2. STATEMENT OF RESULTS

The main goal of this note is to show that Beurling's criterion can be modified to become a necessary and sufficient test for extremal metrics. In fact, we establish a characterization of extremal metrics in the general setting of Fuglede modulus, when  $1 < p < \infty$  and when  $p = 1$ .

**Theorem 1** (Extremal Metrics in  $L^p$ ). *Let  $1 < p < \infty$ . Let  $\mathbf{E}$  be a measure system and let  $\varphi$  be an admissible metric for  $\mathbf{E}$  such that  $\varphi \in L^p(m)$ . Then  $\varphi$  is extremal for the  $p$ -modulus of  $\mathbf{E}$  if and only if*

( $B_p$ ) *There exists a measure system  $\mathbf{F}$  such that*

- (a)  $\text{mod}_p \mathbf{E} \cup \mathbf{F} = \text{mod}_p \mathbf{E}$ ,
- (b)  $\int \varphi d\nu = 1$  for every  $\nu \in \mathbf{F}$ , and
- (c) for all  $f \in L^p(m)$  taking values in  $[-\infty, \infty]$ : if  $\int f d\nu \geq 0$  for all  $\nu \in \mathbf{F}$  then  $\int f \varphi^{p-1} dm \geq 0$ .

**Theorem 2** (Extremal Metrics in  $L^1$ ). *Let  $\mathbf{E}$  be a measure system and let  $\varphi$  be an admissible metric for  $\mathbf{E}$  such that  $\varphi \in L^1(m)$ . Then  $\varphi$  is extremal for the 1-modulus of  $\mathbf{E}$  if and only if*

- (B<sub>1</sub>) *There exists a measure system  $\mathbf{F}$  such that*
  - (a)  $\text{mod}_1 \mathbf{E} \cup \mathbf{F} = \text{mod}_1 \mathbf{E}$ ,
  - (b)  $\int \varphi d\nu = 1$  for every  $\nu \in \mathbf{F}$ , and
  - (c) *for all  $f \in L^1(m)$  taking values in  $[-\infty, \infty]$  such that  $\varphi(x) = 0 \Rightarrow f(x) \geq 0$ : if  $\int f d\nu \geq 0$  for all  $\nu \in \mathbf{F}$  then  $\int f dm \geq 0$ .*

*Remark 3.* We label the conditions (B<sub>p</sub>) in Theorems 1 and 2 in honor of Beurling. When  $p = 2$  and  $\mathbf{F} \subset \mathbf{E}$ , (a) holds vacuously and (B<sub>2</sub>) is Beurling's criterion.

*Remark 4.* In Theorems 1 and 2, if  $\varphi$  is extremal for  $\text{mod}_p \mathbf{E}$ , then there exists  $\mathbf{F}$  satisfying (B<sub>p</sub>) such that for every  $\nu \in \mathbf{F}$  there exist  $\mu \in \mathbf{E}$  and  $0 < c \leq 1$  such that  $\nu = c\mu$ .

The proofs of Theorems 1 and 2 will be given in sections 3 and 4, respectively. We now demonstrate their use in a simple, yet enlightening example, which shows the varied behavior of the  $p$ -modulus for different values of  $p$ .

*Example 5.* Assume that  $A \in \mathcal{M}$  and  $0 < m(A) < \infty$ . Let  $\mathbf{E}_A = \{m \llcorner A\}$  where  $m \llcorner A$  denotes the measure  $m$  restricted to the set  $A$ . Then

$$\text{mod}_p \mathbf{E}_A = \begin{cases} \inf\{m(B)^{1-p} : B \subset A, m(B) > 0\}, & \text{if } 0 < p \leq 1, \\ m(A)^{1-p}, & \text{if } 1 \leq p < \infty. \end{cases}$$

This will be checked in four steps.

Let  $1 < p < \infty$ . We will show that  $\varphi_A = m(A)^{-1} \chi_A$  is extremal for  $\text{mod}_p \mathbf{E}_A$ , and hence,  $\text{mod}_p \mathbf{E}_A = \int_A m(A)^{-p} dm = m(A)^{1-p}$ . Clearly  $\varphi_A \in L^p(m)$  and  $\varphi_A$  is admissible for  $\mathbf{E}_A$ . Let us check that (B<sub>p</sub>) holds with  $\mathbf{F} = \mathbf{E}_A$ . Conditions (a) and (b) hold immediately. For condition (c),  $\int f \varphi_A^{p-1} dm = m(A)^{1-p} \int_A f dm \geq 0$  whenever  $\int f d(m \llcorner A) \geq 0$ . Thus,  $\varphi_A$  is extremal for  $\text{mod}_p \mathbf{E}_A$ , by Theorem 1.

The case  $p = 1$  is similar, except that there is no longer a unique extremal metric. Let  $B \subset A$  be any subset such that  $m(B) > 0$  and let  $\varphi_B = m(B)^{-1} \chi_B$ . Then  $\varphi_B \in L^1(m)$  and  $\varphi_B$  is admissible for  $\mathbf{E}_A$ . We will check that (B<sub>1</sub>) holds with  $\mathbf{F} = \mathbf{E}_A$ . Conditions (a) and (b) are immediate. To verify condition (c) of (B<sub>1</sub>), assume that  $f \in L^1(m)$  takes values in  $[-\infty, \infty]$ ,  $f(x) \geq 0$  whenever  $\varphi_B(x) = 0$  and  $\int f d(m \llcorner A) \geq 0$ . Then  $\int f dm = \int_{A^c} f dm + \int_A f dm \geq 0$ , where the first term is non-negative since  $\varphi_B(x) = 0$  on  $A^c$ . Thus,  $\varphi_B$  is extremal for  $\text{mod}_1 \mathbf{E}_A$ , by Theorem 2, so that  $\text{mod}_1 \mathbf{E}_A = \int \varphi_B dm = 1$  for every  $B \subset A$  with  $m(B) > 0$ .

Next let  $0 < p < 1$  and suppose that  $A$  has subsets of arbitrarily small positive measure. Then we can find a sequence of subsets  $B_k \subset A$  with  $m(B_k) > 0$  such that  $\lim_{k \rightarrow \infty} m(B_k) = 0$ . The normalized characteristic functions  $\varphi_k = m(B_k)^{-1} \chi_{B_k}$  are admissible for  $\mathbf{E}_A$ . Hence  $\text{mod}_p \mathbf{E}_A \leq \int \varphi_k^p dm = m(B_k)^{1-p} \rightarrow 0$  as  $k \rightarrow \infty$ , since  $0 < p < 1$ . Therefore,  $\text{mod}_p \mathbf{E}_A = 0 = \inf\{m(B)^{1-p} : B \subset A, m(B) > 0\}$ . However, there is no extremal metric for  $\text{mod}_p \mathbf{E}_A$ , because no function  $\psi \geq 0$  satisfies  $\int \psi d(m \llcorner A) \geq 1$  and  $\int \psi^p dm = 0$  simultaneously.

Finally, let  $0 < p < 1$ , but suppose that  $A$  does not possess subsets of arbitrarily small positive measure. Then  $m \llcorner A = c_1 \delta_{A_1} + \cdots + c_k \delta_{A_k}$  is a linear combination of atomic measures, where each atom  $A_i \in \mathcal{M}$  and  $A_i \cap A_j = \emptyset$  whenever  $i \neq j$ . By relabeling, we may assume that  $0 < c_1 \leq c_j$  for all  $2 \leq j \leq k$ . Let  $\rho \geq 0$  be an admissible metric for  $\mathbf{E}_A$  such that  $\int \rho d(m \llcorner A) = 1$ . (Here we can ask for equality, because  $\mathbf{E}_A$  consists of one element.) Define  $\delta_j = \rho(A_j) c_j$  for all  $j$ . Then  $\sum_{j=1}^k \delta_j = \sum_{j=1}^k (\delta_j / c_j) c_j = \int \rho d(m \llcorner A) = 1$ . Thus,  $0 \leq \delta_j \leq 1$  for all  $j$ , and

$$\int \rho^p dm \geq \sum_{j=1}^k (\delta_j / c_j)^p c_j = \sum_{j=1}^k \delta_j^p c_j^{1-p} \geq \sum_{j=1}^k \delta_j c_1^{1-p} = c_1^{1-p}.$$

Since the lower bound  $\int \rho^p dm \geq c_1^{1-p}$  is obtained by the metric  $\rho = m(A_1)^{-1} \chi_{A_1}$ , we conclude that  $\text{mod}_p \mathbf{E}_A = m(A_1)^{1-p} = \inf \{m(B)^{1-p} : B \subset A, m(B) > 0\}$ .

*Remark 5.* With the same notation as in the previous example,  $\varphi_A = m(A)^{-1} \chi_A$  also satisfies condition  $(B_p)$  with  $\mathbf{F} = \mathbf{E}_A$  for  $0 < p < 1$ . But  $\text{mod}_p \mathbf{E}_A \neq \int \varphi_A^p dm$  when  $0 < p < 1$  unless  $m \llcorner A = c \delta_A$ . Thus, Example 5 shows that condition  $(B_p)$  from Theorem 1 is not a sufficient test for extremal metrics when  $0 < p < 1$ .

The characterizations of extremal metrics for the  $p$ -modulus of measure systems in Theorems 1 and 2 also hold for curve families in  $\mathbb{R}^n$ . In particular, assuming that an extremal metric for  $\mathbf{E} = \{\tilde{\gamma} : \gamma \in \Gamma\}$  exists, one can find a measure system  $\mathbf{F}$  satisfying condition  $(B_p)$  such that  $\mathbf{F}$  is also associated to a family of curves in Euclidean space.

**Corollary 1** (Extremal Metrics in  $L^p$  for Curves). *Let  $1 < p < \infty$ . Let  $\Gamma$  be a curve family in  $\mathbb{R}^n$  and let  $\varphi$  be an admissible metric for  $\Gamma$  such that  $\varphi \in L^p(\mathbb{R}^n)$ . Then  $\varphi$  is extremal for the  $p$ -modulus of  $\Gamma$  if and only if*

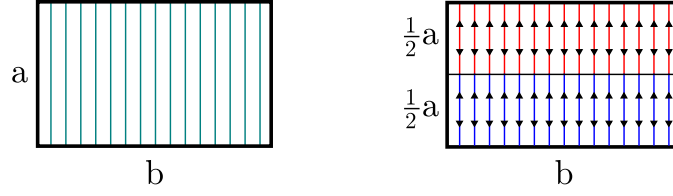
- ( $B'_p$ ) *There exists a curve family  $\Gamma'$  in  $\mathbb{R}^n$  such that*
  - (a)  $\text{mod}_p \Gamma \cup \Gamma' = \text{mod}_p \Gamma$ ,
  - (b)  $\int_\gamma \varphi ds = 1$  for every  $\gamma \in \Gamma'$ , and
  - (c) *for all  $f \in L^p(\mathbb{R}^n)$  taking values in  $[-\infty, \infty]$ : if  $\int_\gamma f ds \geq 0$  for all  $\gamma \in \Gamma'$  then  $\int_{\mathbb{R}^n} f \varphi^{p-1} \geq 0$ .*

**Corollary 2** (Extremal Metrics in  $L^1$  for Curves). *Let  $\Gamma$  be a curve family in  $\mathbb{R}^n$  and let  $\varphi$  be an admissible metric for  $\Gamma$  such that  $\varphi \in L^1(\mathbb{R}^n)$ . Then  $\varphi$  is extremal for the 1-modulus of  $\Gamma$  if and only if*

- ( $B'_1$ ) *There exists a curve family  $\Gamma'$  in  $\mathbb{R}^n$  such that*
  - (a)  $\text{mod}_1 \Gamma \cup \Gamma' = \text{mod}_1 \Gamma$ ,
  - (b)  $\int_\gamma \varphi ds = 1$  for every  $\gamma \in \Gamma'$ , and
  - (c) *for all  $f \in L^1(\mathbb{R}^n)$  taking values in  $[-\infty, \infty]$  such that  $\varphi(x) = 0 \Rightarrow f(x) \geq 0$ : if  $\int_\gamma f ds \geq 0$  for all  $\gamma \in \Gamma'$  then  $\int_{\mathbb{R}^n} f \geq 0$ .*

*Remark 6.* In Corollaries 1 and 2, if  $\varphi$  is extremal for  $\text{mod}_p \Gamma$ , then there exists  $\Gamma'$  satisfying  $(B'_p)$  such that every curve  $\gamma' \in \Gamma'$  is a subcurve of some curve  $\gamma \in \Gamma$ .

The auxiliary curve family  $\Gamma'$  that is required to test condition  $(B'_p)$  is not unique. In the next example, we exhibit disjoint curve families  $\Gamma_0$  and  $\Gamma_1$  such that condition  $(B'_2)$  holds with the auxiliary curve family  $\Gamma' = \Gamma_i$ ,  $i = 0, 1$ .

FIGURE 2. Curve families  $\Gamma_0$  and  $\Gamma_1$  in Example 6

*Example 6.* Let  $R$  be a rectangle with side lengths  $a \leq b$ , and let  $\Gamma$  and  $\Gamma_0$  be the curve families from Example 2. Above we showed that condition  $(B'_2)$  (i.e. Beurling's criterion) holds for  $\Gamma$  and  $\varphi = \frac{1}{a}\chi_R$  using the auxiliary curve family  $\Gamma' = \Gamma_0 \subset \Gamma$ . Thus,  $\text{mod}_2 \Gamma = \int_R (1/a)^2 = b/a$  by Corollary 1. Alternatively let  $\Gamma_1$  be the curve family described as follows. For each  $\gamma_0 \in \Gamma_0$  (teal curve in Figure 2), there correspond exactly two curves  $\gamma'_0$  and  $\gamma''_0$  in  $\Gamma_1$  (red and blue curves). If the curve  $\gamma_0 = [P, Q]$ , then the curves  $\gamma'_0$  and  $\gamma''_0$  are given by

$$\gamma'_0 = \left[ P, \frac{P+Q}{2} \right] \sqcup \left[ \frac{P+Q}{2}, P \right] \quad \text{and} \quad \gamma''_0 = \left[ Q, \frac{P+Q}{2} \right] \sqcup \left[ \frac{P+Q}{2}, Q \right]$$

where  $\sqcup$  denotes concatenation. In other words, each curve in  $\Gamma_1$  travels along a straight path starting at and perpendicular to an edge of side length  $b$ ; half-way across to the other side, the curve reverses direction and returns to its starting point. We now check that  $(B'_2)$  holds for  $\Gamma$  and  $\varphi$  with  $\Gamma' = \Gamma_1$ . A quick computation shows that  $\int_\gamma \varphi ds = 1$  for all  $\gamma \in \Gamma_1$ . Hence condition (b) holds. For (a), we have  $\text{mod}_2 \Gamma \leq \text{mod}_2 \Gamma \cup \Gamma_1 \leq \int_{\mathbb{R}^2} \varphi^2 = \text{mod}_2 \Gamma$  since  $\varphi$  is admissible for  $\Gamma \cup \Gamma_1$  and since (we already know that)  $\varphi$  is extremal for  $\Gamma$ . It remains to check (c). If  $f \in L^2(\mathbb{R}^2)$  and  $\int_\gamma f ds \geq 0$  for all  $\gamma \in \Gamma_1$ , then

$$\begin{aligned} 2 \int_{\mathbb{R}^2} f \varphi &= \frac{2}{a} \int_R f = \frac{2}{a} \int_0^b \int_{\gamma_0(t)} f ds dt = \frac{1}{a} \int_0^b \int_{\gamma_0(t) \sqcup \gamma'_0(t)} f ds dt \\ &= \frac{1}{a} \int_0^b \int_{\gamma'_0(t) \sqcup \gamma''_0(t)} f ds dt = \frac{1}{a} \int_0^b \int_{\gamma'_0(t)} f ds dt + \frac{1}{a} \int_0^b \int_{\gamma''_0(t)} f ds dt \geq 0. \end{aligned}$$

Thus, condition (c) holds too, and we have reached the end of the example.

*Remark 7.* It is of course possible to specialize Theorems 1 and 2 to other settings. For instance, a reader familiar with analysis on metric spaces will have no difficulty adapting Corollaries 1 and 2 to the metric space setting. In [4, §11], Bonk notes that Beurling's criterion can be adapted to produce a sufficient test for extremal metrics for the transboundary modulus of a curve family in the Riemann sphere. Using Theorem 1 and the proof of Corollary 1, one can also formulate a necessary and sufficient test for extremal metrics for transboundary modulus.

So far we have found a characterization of extremal metrics for the  $p$ -modulus of a measure system or curve family when  $1 \leq p < \infty$ . A related problem is to identify those metrics which are extremal for the  $p$ -modulus of some measure system or curve family. The next result gives a solution to this problem for measure systems.

**Theorem 3.** *If  $\varphi : X \rightarrow [0, \infty]$  is a metric and  $\varphi < \infty$   $m$ -a.e., then  $\varphi$  is extremal for the  $p$ -modulus of  $\mathbf{E}_\varphi = \{\mu \text{ defined on } \mathcal{M} : \int \varphi d\mu \geq 1\}$  for all  $0 < p < \infty$ .*

*Proof.* Let  $\mathcal{A}$  be the collection of atoms in  $\mathcal{M}$ . Note that the scaled atomic measure  $\mu_A = \varphi(A)^{-1}\delta_A \in \mathbf{E}_\varphi$  for all  $A \in \mathcal{A}$  such that  $0 < \varphi(A) < \infty$ . Let  $\psi$  be an admissible metric for  $\mathbf{E}_\varphi$ . Then  $\psi(A)/\varphi(A) = \int \psi d\mu_A \geq 1$  when  $0 < \varphi(A) < \infty$ . Also,  $\psi(A) \geq \varphi(A)$  when  $\varphi(A) = 0$ . Thus, if  $\varphi < \infty$   $m$ -a.e., then  $\psi \geq \varphi$   $m$ -a.e., and  $\int \psi^p dm \geq \int \varphi^p dm$  for all  $0 < p < \infty$ . Therefore,  $\text{mod}_p \mathbf{E}_\varphi = \int \varphi^p dm$  for all  $0 < p < \infty$ .  $\square$

We can establish a similar result for curve families in  $\mathbb{R}^n$ . The basic philosophy, suggested by the proof of Theorem 3, is that one needs to approximate the measures  $\delta_x$  at points where  $\varphi(x) > 0$  by sequences of curves. See section 6 for details.

**Theorem 4.** *If  $\varphi : \mathbb{R}^n \rightarrow [0, \infty]$  is Borel, then  $\varphi$  is extremal for the  $p$ -modulus of  $\Gamma_\varphi = \{\text{curve } \gamma \text{ in } \mathbb{R}^n : \int_\gamma \varphi ds \geq 1\}$  for all  $1 \leq p < \infty$  such that  $0 < \int_{\mathbb{R}^n} \varphi^p < \infty$ .*

The plan for the remainder of the note is as follows. In the next two sections, we prove the characterizations of extremal metrics for the  $p$ -modulus of a measure system from above, in the cases  $1 < p < \infty$  (section 3) and  $p = 1$  (section 4). Then we turn our attention to extremal metrics for families of curves in  $\mathbb{R}^n$ . In section 5, we show how the proofs of Theorems 1 and 2 must be modified to obtain Corollaries 1 and 2. Finally, we give the proof of Theorem 4 in section 6.

### 3. PROOF OF THEOREM 1 (EXTREMAL METRICS IN $L^p$ )

Let  $1 < p < \infty$ . Let  $\mathbf{E}$  be a measure system and let  $\varphi$  be an admissible metric for  $\mathbf{E}$  such that  $\varphi \in L^p(m)$ . If  $\varphi = 0$   $m$ -a.e., then  $\varphi$  is extremal for the  $p$ -modulus of  $\mathbf{E}$  and condition  $(B_p)$  holds with  $\mathbf{F} = \emptyset$ . Thus, we assume that  $0 < \int \varphi^p dm < \infty$ .

We shall start with the proof that condition  $(B_p)$  implies that  $\varphi$  is extremal. Suppose that  $(B_p)$  holds for some measure system  $\mathbf{F}$  satisfying (a), (b) and (c). Since the metric  $\varphi$  is admissible for  $\mathbf{E}$ ,  $\varphi$  is also admissible for  $\mathbf{E} \cup \mathbf{F}$ , by (b). Let  $\psi$  be a competing admissible metric for  $\text{mod}_p \mathbf{E} \cup \mathbf{F}$ , so that  $\int \psi^p dm \leq \int \varphi^p dm < \infty$ . Then  $\int \psi d\nu \geq 1 = \int \varphi d\nu$  for all  $\nu \in \mathbf{F}$ , by (b). Hence  $f = \psi - \varphi \in L^p(m)$  and  $\int f d\nu \geq 0$  for all  $\nu \in \mathbf{F}$ . By (c), we conclude that  $\int (\psi - \varphi)\varphi^{p-1} \geq 0$ . Then

$$(1) \quad \int \varphi^p dm \leq \int \psi \varphi^{p-1} dm \leq \left( \int \psi^p dm \right)^{1/p} \left( \int \varphi^p dm \right)^{(p-1)/p}$$

where the second inequality is Hölder's inequality. Since  $0 < \int \varphi^p dm < \infty$ , we get that  $\int \varphi^p dm \leq \int \psi^p dm$ . Thus,  $\varphi$  is extremal for the  $p$ -modulus of  $\mathbf{E} \cup \mathbf{F}$ . Finally,  $\text{mod}_p \mathbf{E} \leq \int \varphi^p dm = \text{mod}_p \mathbf{E} \cup \mathbf{F} = \text{mod}_p \mathbf{E}$ , by (a). Therefore,  $\varphi$  is extremal for the  $p$ -modulus of  $\mathbf{E}$ .

For the reverse direction, we require a short lemma.

**Lemma 1.** *Let  $1 < p < \infty$ . If  $\varphi, f \in L^p(m)$  take values in  $[-\infty, \infty]$  and  $\varphi \geq 0$ , then*

$$\int [(\varphi + \varepsilon f)^+]^p dm = \int_{\{\varphi + \varepsilon f > 0\}} [\varphi^p + p\varepsilon f \varphi^{p-1}] dm + o(\varepsilon) \cdot \varepsilon$$

where  $o(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

*Proof.* Let  $1 < p < \infty$  and let  $\varphi, f \in L^p(m)$  be given. Assume that the functions  $\varphi$  and  $f$  take values in  $[-\infty, \infty]$  and  $\varphi \geq 0$ . Fix  $\varepsilon \neq 0$  and set  $P = \{\varphi + \varepsilon f > 0\}$ . By the mean value theorem, for all  $x \in P$  such that  $\varphi(x)$  and  $f(x)$  are both finite,



there exists  $\delta = \delta(x)$  between 0 and  $\varepsilon$  such that  $(\varphi + \varepsilon f)^p - \varphi^p = [p(\varphi + \delta f)^{p-1} f] \varepsilon$ . In particular, this holds at  $m$ -a.e.  $x \in P$ , since  $\varphi, f \in L^p(m)$ . Hence

$$\int_P (\varphi + \varepsilon f)^p dm = \int_P [\varphi^p + p\varepsilon f \varphi^{p-1}] dm + \varepsilon \int_P p f [(\varphi + \delta f)^{p-1} - \varphi^{p-1}] dm.$$

The lemma follows, because the second integral in the displayed equation vanishes as  $\varepsilon \rightarrow 0$  by the dominated convergence theorem.  $\square$

Now suppose that  $\varphi$  is extremal for the  $p$ -modulus of  $\mathbf{E}$ . Break  $\mathbf{E} = \mathbf{E}_0 \cup \mathbf{E}_\infty$  into a union of two measure systems where  $\mathbf{E}_0 = \{\mu \in \mathbf{E} : 1 \leq \int \varphi d\mu < \infty\}$  and  $\mathbf{E}_\infty = \{\mu \in \mathbf{E} : \int \varphi d\mu = \infty\}$ . Since  $\varphi \in L^p(m)$ , we have  $\text{mod}_p \mathbf{E}_\infty = 0$ , because  $\varepsilon \varphi$  is admissible for  $\mathbf{E}_\infty$  for all  $\varepsilon > 0$ . It follows that  $\text{mod}_p \mathbf{E}_0 = \text{mod}_p \mathbf{E} = \int \varphi^p dm$ ; that is,  $\varphi$  is extremal for the  $p$ -modulus of  $\mathbf{E}_0$ , as well. Moreover,  $\mathbf{E}_0$  is nonempty, since  $\text{mod}_p \mathbf{E}_0 > 0$ . Recall that we want to show that condition  $(B_p)$  holds. Assign  $\mathbf{F}$  to be the family of all measures  $\nu$  defined on  $\mathcal{M}$  such that  $\int \varphi d\nu = 1$ . Thus, (b) is satisfied by the definition of  $\mathbf{F}$ . To verify (a), simply note that

$$\text{mod}_p \mathbf{E} \leq \text{mod}_p \mathbf{E} \cup \mathbf{F} \leq \int \varphi^p dm = \text{mod}_p \mathbf{E},$$

since  $\varphi$  is admissible for  $\mathbf{E} \cup \mathbf{F}$  and  $\varphi$  is extremal for the  $p$ -modulus of  $\mathbf{E}$ . It remains to establish (c). Assume that  $f \in L^p(m)$  takes values in  $[-\infty, \infty]$  and  $\int f d\nu \geq 0$  for every  $\nu \in \mathbf{F}$ . Then for all  $\varepsilon > 0$  the metric  $\varphi_\varepsilon = (\varphi + \varepsilon f)^+ \geq 0$  belongs to  $L^p(m)$  and  $\int \varphi_\varepsilon d\nu \geq \int (\varphi + \varepsilon f) d\nu \geq \int \varphi d\nu = 1$  for every  $\nu \in \mathbf{F}$ . If  $\mu \in \mathbf{E}_0$ , then there exists  $0 < c \leq 1$  such that  $c\mu \in \mathbf{F}$  so that  $\int \varphi_\varepsilon d\mu \geq c \int \varphi_\varepsilon d\mu = \int \varphi_\varepsilon d(c\mu) \geq 1$ . Hence the metric  $\varphi_\varepsilon$  is also admissible for  $\mathbf{E}_0$ . Thus,

$$\int \varphi^p dm = \text{mod}_p \mathbf{E}_0 \leq \int \varphi_\varepsilon^p dm = \int [(\varphi + \varepsilon f)^+]^p dm.$$

Then, Lemma 1 gives  $\int \varphi^p dm \leq \int_{P_\varepsilon} [\varphi^p + p\varepsilon f \varphi^{p-1}] dm + o(\varepsilon) \cdot \varepsilon$ , where the set  $P_\varepsilon = \{\varphi + \varepsilon f > 0\}$  and  $o(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . It follows that

$$p\varepsilon \int_{P_\varepsilon} f \varphi^{p-1} dm \geq \int_{X \setminus P_\varepsilon} \varphi^p dm - o(\varepsilon) \cdot \varepsilon \geq -o(\varepsilon) \cdot \varepsilon.$$

Dividing through by  $p\varepsilon$  and letting  $\varepsilon \rightarrow 0+$ , we obtain

$$\int f \varphi^{p-1} dm = \lim_{\varepsilon \rightarrow 0+} \int_{P_\varepsilon} f \varphi^{p-1} dm \geq 0,$$

by the dominated convergence theorem. Therefore, condition  $(B_p)$  holds if  $\varphi$  is extremal for the  $p$ -modulus of  $\mathbf{E}$ .

#### 4. PROOF OF THEOREM 2 (EXTREMAL METRICS IN $L^1$ )

Let  $\mathbf{E}$  be a measure system and let  $\varphi$  be an admissible metric for  $\mathbf{E}$  such that  $\varphi \in L^1(m)$ . If  $\varphi = 0$   $m$ -a.e., then  $\varphi$  is extremal for the 1-modulus of  $\mathbf{E}$  and condition  $(B_1)$  holds with  $\mathbf{F} = \emptyset$ . Thus, we assume that  $0 < \int \varphi dm < \infty$ .

Suppose that condition  $(B_1)$  holds. Let  $\psi$  be an admissible metric for  $\mathbf{E} \cup \mathbf{F}$  with  $\psi \in L^1(m)$ . Then  $\int \psi d\nu \geq 1 = \int \varphi d\nu$  for every  $\nu \in \mathbf{F}$ , by (b). Hence the function  $f = \psi - \varphi \in L^1(m)$  takes values in  $[-\infty, \infty]$ ,  $\int f d\nu \geq 0$  for all  $\nu \in \mathbf{F}$  and  $f(x) \geq 0$  whenever  $\varphi(x) = 0$ . Since  $f$  satisfies the hypothesis of (c), we obtain  $\int (\psi - \varphi) dm \geq 0$ . That is,  $\int \varphi dm \leq \int \psi dm$ , for every admissible metric  $\psi$ . Thus,  $\varphi$  is extremal for the 1-modulus of  $\mathbf{E} \cup \mathbf{F}$ . It follows that  $\text{mod}_1 \mathbf{E} \leq \int \varphi dm = \text{mod}_1 \mathbf{E} \cup \mathbf{F} = \text{mod}_1 \mathbf{E}$ , by (a). Therefore,  $\varphi$  is extremal for the 1-modulus of  $\mathbf{E}$ .

Conversely, suppose that  $\varphi$  is extremal for the 1-modulus of  $E$ . Then  $\varphi$  is also extremal for the 1-modulus of  $\mathbf{E}_0 = \{\mu \in \mathbf{E} : 1 \leq \int \varphi d\mu < \infty\}$ . We want to check that condition  $(B_1)$  holds. Assign  $\mathbf{F}$  to be the family of all measures  $\nu$  defined on  $\mathcal{M}$  such that  $\int \varphi d\nu = 1$ . Then (b) is satisfied automatically. For (a),  $\text{mod}_1 \mathbf{E} \leq \text{mod}_1 \mathbf{E} \cup \mathbf{F} \leq \int \varphi dm = \text{mod}_1 \mathbf{E}$ , since  $\varphi$  is admissible for  $\mathbf{E} \cup \mathbf{F}$  and  $\varphi$  is extremal for the 1-modulus of  $\mathbf{E}$ . It remains to verify (c). Assume that  $f \in L^1(m)$  takes values in  $[-\infty, \infty]$ ,  $\int f d\nu \geq 0$  for all  $\nu \in \mathbf{F}$  and  $f(x) \geq 0$  whenever  $\varphi(x) = 0$ . For all  $\varepsilon > 0$  the metric  $\varphi_\varepsilon = (\varphi + \varepsilon f)^+ \geq 0$  belongs to  $L^1(m)$ , and moreover, satisfies  $\int \varphi_\varepsilon d\nu \geq \int (\varphi + \varepsilon f) d\nu \geq \int \varphi d\nu = 1$  for every  $\nu \in \mathbf{F}$ . Now, for all  $\mu \in \mathbf{E}_0$ , there exists  $0 < c \leq 1$  such that  $c\mu \in \mathbf{F}$ . Thus,  $\int \varphi_\varepsilon d\mu \geq c \int \varphi_\varepsilon d\mu = \int \varphi_\varepsilon d(c\mu) \geq 1$  for all  $\mu \in \mathbf{E}_0$ . This shows that the metric  $\varphi_\varepsilon$  is also admissible for  $\mathbf{E}_0$ , and hence,

$$\int \varphi dm = \text{mod}_1 \mathbf{E}_0 \leq \int \varphi_\varepsilon dm = \int (\varphi + \varepsilon f)^+ dm = \int_{P_\varepsilon} (\varphi + \varepsilon f) dm,$$

where  $P_\varepsilon = \{\varphi + \varepsilon f > 0\}$ . This yields  $\int_{P_\varepsilon} f dm \geq \varepsilon^{-1} \int_{X \setminus P_\varepsilon} \varphi dm \geq 0$  for all  $\varepsilon > 0$ . As  $\varepsilon \rightarrow 0+$ , the characteristic functions  $\chi_{P_\varepsilon}$  converge  $m$ -a.e. to the function  $\chi_P$  where  $P = \{\varphi > 0\} \cup \{\varphi = 0, f > 0\}$  (convergence at  $x \in X$  fails if  $f(x) = -\infty$ ). Therefore,  $\int_P f dm \geq 0$ , and because we assumed that  $f(x) \geq 0$  whenever  $\varphi(x) = 0$ , we obtain  $\int f dm = \int_P f dm + \int_{\{\varphi=0, f=0\}} f dm \geq 0$ , as well. This completes the proof that condition  $(B_1)$  holds whenever  $\varphi$  is extremal for the 1-modulus of  $\mathbf{E}$ .

## 5. MODIFICATION FOR CURVES FAMILIES IN $\mathbb{R}^n$

Let  $1 \leq p < \infty$ . Let  $\Gamma \subset \mathbb{R}^n$  be a curve family and let  $\varphi$  be an extremal metric for the  $p$ -modulus of  $\Gamma$  such that  $0 < \int_{\mathbb{R}^n} \varphi^p < \infty$ . Then the metric  $\varphi$  is also extremal for the  $p$ -modulus of  $\Gamma_0 = \{\gamma \in \Gamma : 1 \leq \int_\gamma \varphi ds < \infty\}$ . We want to check that condition  $(B'_p)$  holds. Assign  $\Gamma'$  to be the family of all curves  $\gamma$  in  $\mathbb{R}^n$  such that  $\int_\gamma \varphi ds = 1$ . Then (b) holds by definition. To show (a), simply note that  $\text{mod}_p \Gamma \leq \text{mod}_p \Gamma \cup \Gamma' \leq \int_{\mathbb{R}^n} \varphi^p = \text{mod}_p \Gamma$ , because  $\varphi$  is admissible for  $\Gamma \cup \Gamma'$  and  $\varphi$  is extremal for the  $p$ -modulus of  $\Gamma$ . It remains to verify (c). Assume that  $f \in L^p(\mathbb{R}^n)$  takes values in  $[-\infty, \infty]$  and  $\int_\gamma f ds \geq 0$  for all  $\gamma \in \Gamma'$ . In the case  $p = 1$ , also assume that  $f(x) \geq 0$  whenever  $\varphi(x) = 0$ . For all  $\varepsilon > 0$ , the metric  $\varphi_\varepsilon = (\varphi + \varepsilon f)^+ \geq 0$  belongs to  $L^p(\mathbb{R}^n)$ . Moreover,  $\int_\gamma \varphi_\varepsilon ds \geq \int_\gamma (\varphi + \varepsilon f) ds \geq \int_\gamma \varphi ds = 1$  for all  $\gamma \in \Gamma'$ . If  $\gamma \in \Gamma_0$ , then

$$1 \leq \int_\gamma \varphi ds = \sum_i \int_{a_i}^{b_i} \varphi(\gamma_i(t)) |\gamma'_i(t)| dt < \infty.$$

Since each term in the line integral is non-negative and finite, the function

$$c \in [a_i, b_i] \mapsto \int_{a_i}^c \varphi(\gamma_i(t)) |\gamma'_i(t)| dt$$

is continuous for each  $i$ . Hence we can pick  $c_i \in [a_i, b_i]$  for all  $i$  in such a way that the subcurve  $\gamma_1 = \bigsqcup_i \gamma([a_i, c_i])$  of  $\gamma$  satisfies  $\int_{\gamma_1} \varphi ds = 1$ . This means that  $\gamma_1 \in \Gamma'$ . Thus,  $\int_{\gamma_1} \varphi_\varepsilon ds \geq \int_{\gamma_1} \varphi ds \geq 1$ . This shows that  $\varphi_\varepsilon$  is also admissible for  $\Gamma_0$ . Hence

$$\int_{\mathbb{R}^n} \varphi^p = \text{mod}_p \Gamma_0 \leq \int_{\mathbb{R}^n} \varphi_\varepsilon^p = \int_{\mathbb{R}^n} [(\varphi + \varepsilon f)^+]^p.$$

To finish checking (c), one can now proceed as above. Follow the argument from section 3, when  $1 < p < \infty$ , and follow the argument from section 4, when  $p = 1$ .

## 6. PROOF OF THEOREM 4

Suppose that  $\varphi : \mathbb{R}^n \rightarrow [0, \infty]$  is a Borel function and let  $\Gamma_\varphi$  be the family of all curves  $\gamma$  in  $\mathbb{R}^n$  such that  $\int_\gamma \varphi ds \geq 1$ . Fix any  $1 \leq p < \infty$  such that  $0 < \int_{\mathbb{R}^n} \varphi^p < \infty$ . We want to show that  $\varphi$  is extremal for the  $p$ -modulus of  $\Gamma_\varphi$ . For each  $y \in \mathbb{R}^n$  let  $\ell_y = y + \mathbb{R}e_1 \cong \mathbb{R}$  denote the line through  $y$  parallel to the direction  $e_1 = (1, 0, \dots, 0)$ . By Fubini's theorem,  $\varphi \in L^p(\ell_y)$ ,  $y = (0, \bar{y})$  for  $H^{n-1}$ -a.e.  $\bar{y} \in \mathbb{R}^{n-1}$ . In particular, we also have  $\varphi \in L^1_{\text{loc}}(\ell_y)$ ,  $y = (0, \bar{y})$  for  $H^{n-1}$ -a.e.  $\bar{y} \in \mathbb{R}^{n-1}$ . Here, as above and as below,  $H^s$  denote  $s$ -dimensional Hausdorff measure. Below  $|I|$  denotes the diameter of an interval  $I$ .

**Lemma 2.** *Suppose that  $\varphi \in L^1_{\text{loc}}(\ell_y)$ . Then, for  $H^1$ -a.e.  $x \in \ell_y$  such that  $\varphi(x) > 0$ , there exist a sequence of positive integers  $n_k = n_k(x) \rightarrow \infty$  and a sequence intervals  $I_k = I_k(x) \subset \ell_y$  centered at  $x$  with  $|I_k| \rightarrow 0$  such that  $\int_{I_k} \varphi dt = 1/n_k$  for all  $k$ .*

*Proof.* Define the function  $g_x(r) = \int_{-r}^r \varphi(x + te_1) dt$  for all  $x \in \ell_y$  and  $r \geq 0$ . Then  $\lim_{r \rightarrow 0+} g_x(r)/2r = \varphi(x)$  for  $H^1$ -a.e.  $x \in \ell_y$ , by the Lebesgue differentiation theorem. Hence for  $H^1$ -a.e.  $x \in \ell_y$  such that  $\varphi(x) > 0$ , there exists  $r_0 = r_0(x) > 0$  such that  $0 < g_x(r) < \infty$  for all  $0 < r \leq r_0$ . Since  $g_x|_{[0, r_0]}$  is continuous and  $g_x(0) = 0$ , we can find a sequence of integers  $n_k = n_k(x)$  and a sequence of radii  $r_k = r_k(x) \rightarrow 0$  such that  $g_x(r_k) = 1/n_k$ . Then  $I_k = I_k(x) = x + [-r, r]e_1 \subset \ell_y$  is a sequence of intervals with the desired property.  $\square$

Let  $E \subset \mathbb{R}^n$  be the set of points  $x \in \mathbb{R}^n$  where the conclusion of Lemma 2 holds, i.e.  $x \in E$  if and only if there exists a sequence of positive integers  $n_k = n_k(x) \rightarrow \infty$  and a sequence of intervals  $I_k = I_k(x) \subset \ell_x$  centered at  $x$  with  $|I_k| \rightarrow 0$  such that  $\int_{I_k} \varphi dt = 1/n_k$ . By Fubini's theorem and Lemma 2, we have  $x \in E$  for a.e.  $x \in \mathbb{R}^n$  such that  $\varphi(x) > 0$ . We define a curve family  $\Gamma' \subset \Gamma_\varphi$  as follows. Choose one pair of sequences  $(n_k(x))_{k=1}^\infty$  and  $(I_k(x))_{k=1}^\infty$  for each  $x \in E$ . Then, for each  $x \in E$  and  $k \geq 1$ , define a curve  $\gamma_k(x) = \bigsqcup_{i=1}^{n_k} I_k(x)$ , i.e. let  $\gamma_k(x)$  be a curve which covers the interval  $I_k(x)$  exactly  $n_k(x)$ -times. Set  $\Gamma' = \{\gamma_k(x) : x \in E \text{ and } k \geq 1\}$ .

To prove that  $\varphi$  is extremal for the  $p$ -modulus of  $\Gamma_\varphi$ , it is enough by either Corollary 1 or Corollary 2 (according to whether  $1 < p < \infty$  or  $p = 1$ ) to show that  $(B'_p)$  holds for  $\Gamma'$ . To start note  $\int_{\gamma_k(x)} \varphi ds = n_k(x) \int_{I_k(x)} \varphi dt = n_k(x)/n_k(x) = 1$  for all  $\gamma_k(x) \in \Gamma'$ . This shows that (b) holds. And, since  $\Gamma' \subset \Gamma_\varphi$ , (a) is true too. To verify (c), assume that  $f \in L^p(\mathbb{R}^n)$  take values in  $[-\infty, \infty]$  and  $\int_{\gamma_k(x)} f ds \geq 0$  for all  $\gamma_k(x) \in \Gamma'$ . Moreover, if  $p = 1$ , assume that  $f(x) \geq 0$  whenever  $\varphi(x) = 0$ . By Fubini's theorem and the Lebesgue differentiation theorem,

$$f(x) = \lim_{k \rightarrow \infty} \frac{1}{|I_k(x)|} \int_{I_k(x)} f dt$$

for a.e.  $x \in E$ , and in particular, for a.e.  $x \in \mathbb{R}^n$  such that  $\varphi(x) > 0$ . By assumption,

$$\int_{I_k(x)} f dt = \frac{1}{n_k(x)} \int_{\gamma_k(x)} f ds \geq 0 \quad \text{for all } x \in E \text{ and } k \geq 1.$$

Thus, combining the two displayed equations,  $f(x) \geq 0$  at a.e.  $x \in \mathbb{R}^n$  such that  $\varphi(x) > 0$ . It follows that  $\int_{\mathbb{R}^n} f \varphi^{p-1} \geq 0$ , if  $1 < p < \infty$ , and  $\int_{\mathbb{R}^n} f \geq 0$ , if  $p = 1$ . Hence (c) holds. Therefore,  $(B'_p)$  holds and  $\varphi$  is extremal for the  $p$ -modulus of  $\Gamma_\varphi$ .

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